

A hidden BFKL / XXX $_{-\frac{1}{2}}$ spin chain mapping*

Alberto Romagnoni & Agustín Sabio Vera

Instituto de Física Teórica UAM/CSIC, Nicolás Cabrera 15,
& Universidad Autónoma de Madrid, E-28049 Madrid, Spain

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A new mapping between the BFKL equation and Beisert's representation of the XXX Heisenberg ferromagnet with spin $s = -\frac{1}{2}$ is given. The action of the Hamiltonian operator of a spin chain with $SL(2)$ invariance on a symmetric double copy of a harmonic oscillator excited state is shown to be identical to the action of the BFKL Hamiltonian on the gluon Green function for the azimuthal-angle averaged forward scattering case. A natural mapping between the gluon Green function, discretized in virtuality space, and the double harmonic oscillator excited state emerges.

1 Introduction

In recent years there has been a growing activity concerning the identification of integrable structures in four-dimensional gauge theories. This is mainly due to the interest that this subject has for the anti de Sitter / conformal field theory (AdS/CFT) conjecture [1]. After the seminal works in [2], big progress has been made in the mapping of anomalous dimensions of gauge invariant Wilson operators in super Yang-Mills (SYM) theory to the spectrum of string theory in different backgrounds. A crucial step was to realize that the planar one-loop dilatation operator of $\mathcal{N} = 4$ SYM maps into the Hamiltonian of an integrable spin chain. The problem of finding anomalous dimensions translates then into the diagonalization of the corresponding Hamiltonian, and all the techniques developed for integrable systems become very useful in accomplishing this task. After those first results, the better understanding of the mapping has allowed more general results for larger orders in perturbation theory and for different sectors of the gauge and string theories (for an introduction to the field and a wider bibliography see [3]).

Nonetheless, two-dimensional integrable structures in four-dimensional gauge field theory appeared well before the AdS/CFT conjecture, in the region of high energy scattering in Quantum Chromodynamics (QCD). Non-Abelian gauge theories manifest interesting mathematical properties when they are investigated in terms of high energy scattering amplitudes in the Regge limit. The most remarkable example [4] is the $SL(2, \mathbb{C})$ invariance present in the impact parameter representation of QCD (and $\mathcal{N} = 4$ SYM) elastic scattering amplitudes evaluated in multi-Regge kinematics [5–9]. In this context the Balitsky-Fadin-Kuraev-Lipatov (BFKL) pomeron (with vacuum quantum numbers exchanged in the t -channel) can be interpreted as a bound state of two reggeized gluons where the Hamiltonian has an interesting operator representation [10] with holomorphic separability in coordinate space [11]. The iteration of the BFKL Hamiltonian in the s -channel,

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describing multiple reggeon exchanges in the generalized leading logarithmic approximation, defines the Bartels-Kwiecinski-Praszalowicz (BKP) equation [13, 14] and was found to have a hidden integrability [4, 11, 15], being equivalent to a periodic spin chain of a XXX Heisenberg ferromagnet [16–18]. This was the first example of the existence of integrable systems in QCD. A similar integrable spin chain, an open one this time, has recently been found in kinematical regions of n -point maximally helicity violating (MHV) and planar ($N_c \rightarrow \infty$) amplitudes in $\mathcal{N} = 4$ SYM where Mandelstam cut contributions are maximally enhanced [19]. The importance of Mandelstam cuts in the complex angular momentum plane for $\mathcal{N} = 4$ SYM MHV planar amplitudes was first realized in [20, 21] where corrections to the Bern-Dixon-Smirnov (BDS) iterative ansatz [22] for this class of amplitudes were found for the six-point amplitude at two loops.

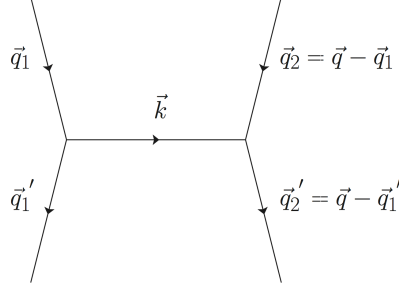
There is an interesting connection between the integrable structures appearing in the calculation of the anomalous dimension of gauge invariant twist (scaling dimension minus Lorentz spin) two operators of spin M in $\mathcal{N} = 4$ SYM, and in the multi-Regge kinematics. As it was shown in [23], the link to the BFKL equation appears upon analytically continuing the anomalous dimension function to complex values of M . In particular, the pomeron corresponds to the first singularity at $M = \omega - 1$, for small ω . The discrepancy between this result and the prediction obtained from the asymptotic Bethe ansatz has been subsequently explained in [24] by the calculation of the corresponding wrapping corrections for the twist two operators.

The aim of the present work is to add new hints to the relation between BFKL physics and integrable systems. In particular, a new mapping between the two-sites one-loop Hamiltonian of the $\mathfrak{sl}(2)$ sector of the $\mathcal{N} = 4$ SYM theory, in the double oscillator picture for operators of a given spin introduced by Beisert in [25], and a matrix version of the forward BFKL equation, after averaging over azimuthal angle, is obtained.

After this brief Introduction to the subject, in Section 2, a general discussion on the BFKL equation is provided, explaining the connection between the non-forward and forward limits. A novel discretization in virtuality space is described in detail, highlighting the role and physical interpretation of the shift and diagonal operators appearing in this representation. The gluon Green function is built in this discrete representation and the limit to the continuum discussed. In Section 3, known facts about Beisert’s representation of the non-compact $\text{SL}(2)$ spin chain are introduced to set the ground for a direct comparison with the BFKL equation, in Section 4. The mapping between the Hamiltonians in both systems and the gluon Green function in BFKL and diagonal states in the $\text{SL}(2)$ spin chain is then transparent. Section 5 is devoted to a discussion of the results and to indicate possible connections with other works. The Conclusions are presented in the final section of the paper.

2 Matrix representation of the forward BFKL equation

In this work the BFKL Hamiltonian is considered directly in two-dimensional transverse momentum space (\vec{k}), where other components have decoupled into an evolution variable (rapidity Y) which plays the role of time [5–9]. The four-point amplitude for off-shell reggeized gluons has the following momentum flow:



With this notation, the BFKL kernel for the pomeron channel has two contributions. The first one corresponds to $\vec{k} = 0$, *i.e.*, there is no propagator in the s -channel and can be written as

$$\begin{aligned} \text{“Reggeized Propagators”} &\simeq g^2 N_c \delta^{(2)}(\vec{q}_1 - \vec{q}_1') \delta^{(2)}(\vec{q}_2 - \vec{q}_2') \\ &\times \left(\int d^2 \vec{r} \frac{\vec{q}_1'^2}{\vec{r}^2 (\vec{q}_1 - \vec{r})^2} + \int d^2 \vec{r} \frac{\vec{q}_2'^2}{\vec{r}^2 (\vec{q}_2 - \vec{r})^2} \right). \end{aligned} \quad (1)$$

The second piece has $\vec{k} \neq 0$ and corresponds to squaring the Lipatov's vertex, *i.e.*,

$$\text{“Emission”} \simeq \delta^{(2)}(\vec{q}_1 + \vec{q}_2 - \vec{q}_1' - \vec{q}_2') \frac{g^2 N_c}{\vec{q}_1^2 \vec{q}_1'^2} \left(\frac{\vec{q}_1^2 \vec{q}_2'^2 + \vec{q}_2^2 \vec{q}_1'^2}{\vec{k}^2} - (\vec{q}_1 + \vec{q}_2)^2 \right). \quad (2)$$

After a Fourier transform of these two expressions, and complexifying the transverse momenta, Lipatov found the very interesting $SL(2, \mathbb{C})$ invariance of this Hamiltonian [4]. In the present work, however, the focus lies on the forward limit, with zero momentum transfer $\vec{q} = 0$. It is noteworthy that in this case the contribution from the “Reggeized Propagators” reads

$$\text{“Reggeized Propagators”} \simeq 2g^2 N_c \delta^{(2)}(\vec{q}_1 - \vec{q}_1') \int d^2 \vec{r} \frac{\vec{q}_1'^2}{\vec{r}^2 (\vec{q}_1 - \vec{r})^2}, \quad (3)$$

while the “Emission” piece simplifies to

$$\text{“Emission”} \simeq 2 \frac{g^2 N_c}{\vec{k}^2} = 2 \frac{g^2 N_c}{(\vec{q}_1 - \vec{q}_1')^2}. \quad (4)$$

It is in this forward case that it truly represents a real emission since now the amplitude corresponds to the $2 \rightarrow 3$ inelastic process. A further simplification is very convenient: to integrate over the azimuthal angle formed by the two transverse momenta \vec{q}_1 and \vec{q}_1' .

Once this is done the BFKL equation for forward scattering can be cast in the simple form

$$\frac{\partial \varphi(Q^2, Y)}{\alpha \partial Y} = \int_0^\infty \frac{dq^2}{|q^2 - Q^2|} \left\{ \varphi(q^2, Y) - \frac{2 \min(q^2, Q^2)}{q^2 + Q^2} \varphi(Q^2, Y) \right\}, \quad (5)$$

where $\alpha = \alpha_s N_c / \pi$, the integration takes place over the gluon virtuality $q^2 (\equiv \vec{q}^2)$ and the correspondence with the previous notation is $\vec{q}_1^2 = Q^2$ and $\vec{q}_1'^2 = Q_0^2$. The term with $\varphi(Q^2, Y)$ corresponds to Eq. (3) and the one with $\varphi(q^2, Y)$ to Eq. (4). Since the forward limit has been taken, $\varphi(Q^2, Y)$ is the cut reggeized gluon four-point function for a given rapidity Y with the initial condition $\varphi(Q^2, Y=0) \sim \delta(Q^2 - Q_0^2)$, and it corresponds to the sum of the squares of the $2 \rightarrow 2 + n$ emissions amplitude over any number n of real gluon emissions.

To find the gluon Green function it is convenient to write Eq. (5) in the form

$$\frac{\partial \varphi(Q^2, Y)}{\alpha \partial Y} = \int_0^1 \frac{dx}{1-x} \left\{ \varphi(x Q^2, Y) + \frac{1}{x} \varphi\left(\frac{Q^2}{x}, Y\right) - 2 \varphi(Q^2, Y) \right\}, \quad (6)$$

and then introduce a Mellin transform to obtain

$$\varphi(Q^2, Y) = \int_{a-i\infty}^{a+i\infty} \frac{d\gamma}{2\pi i} \left(\frac{Q^2}{Q_0^2}\right)^{\gamma-1} e^{\alpha Y \chi(\gamma)}, \quad (7)$$

$$\chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma), \quad (8)$$

with ψ being the digamma function and $0 < a < 1$. It is well known that, for asymptotically large values of the rapidity variable Y , this integral tends to

$$\varphi(Q^2, Y) \simeq \frac{e^{\alpha Y 4 \log 2 - \frac{t}{2} - \frac{t^2}{56\zeta(3)\alpha Y}}}{2\sqrt{14\pi\zeta(3)\alpha Y}}, \quad (9)$$

with $t = \log Q^2/Q_0^2$. This implies the following diffusion equation for the function $\phi = \varphi e^{t/2}/\pi$:

$$\frac{\partial \phi}{\alpha \partial Y} = 4 \log 2 \phi + 14\zeta(3) \frac{\partial^2 \phi}{\partial t^2}, \quad (10)$$

which shows that there exist two different flows for the virtualities of the t -channel gluons in the BFKL ladder: one towards the infrared (IR) and one towards the ultraviolet (UV). These IR/UV flows are symmetric since the eigenvalue function $\chi(\gamma)$ is invariant under $\gamma \leftrightarrow 1 - \gamma$.

It is possible to gain further insight into this picture. For clarity, it is convenient to mix Eq. (5) and Eq. (6) using the representation

$$\frac{\partial \varphi(Q^2, Y)}{\alpha \partial Y} = \int_{-Q^2}^{\infty} \frac{dl^2}{|l^2|} \varphi(l^2 + Q^2, Y) + \Omega \varphi(Q^2, Y), \quad (11)$$

where

$$\Omega = -2 \int_0^1 \frac{dx}{1-x}. \quad (12)$$

The integration region with $l^2 < 0$ corresponds to a virtuality flow into the IR, and that with $l^2 > 0$ into the UV. The term proportional to Ω is diagonal in momentum space since it does not change the virtuality of the original gluon. It is formally infinite and allows for the regularization of the infrared divergence in the integration of the other term. This separation is introduced because it will make explicit interesting properties of the Hamiltonian (or evolution kernel since this is the forward case).

The space of virtualities can be discretized using $l^2 = n \Delta$, $Q^2 = N \Delta$, $dl^2 = \Delta$ and the notation $\phi_n \equiv \varphi(l^2, Y)$. The discrete version of Eq. (11) then reads

$$\frac{\partial \phi_N}{\alpha \partial Y} = \sum_{n=1}^{N-1} \frac{\phi_{N-n}}{n} + \sum_{n=1}^{\infty} \frac{\phi_{N+n}}{n} + \Omega \phi_N. \quad (13)$$

Note that, due to scale invariance, there is no dependence on the “virtuality unit” Δ .

To find a matrix representation of the kernel it is useful to introduce the N -dimensional vector $\vec{\phi} \equiv (\phi_1, \phi_2, \dots, \phi_N)^t$, the extended ∞ -dimensional vector $\vec{\phi}_\infty \equiv (\phi_1, \phi_2, \dots, \phi_N, \dots)^t$ and write Eq. (13) in the form

$$\frac{\partial \vec{\phi}}{\partial Y} = \hat{\mathcal{H}}_N \cdot \vec{\phi}_\infty, \quad (14)$$

with the kernel being the following semi-infinite matrix with N rows and ∞ columns:

$$\hat{\mathcal{H}}_N = \begin{pmatrix} \Omega & 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \dots \\ 1 & \Omega & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \dots \\ \frac{1}{2} & 1 & \Omega & 1 & \frac{1}{2} & \frac{1}{3} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{N-1} & \frac{1}{N-2} & \dots & 1 & \Omega & 1 & \dots \end{pmatrix}. \quad (15)$$

In terms of components this is equivalent to

$$(\hat{\mathcal{H}}_N)_{i,j} = \sum_{n=1}^{N-1} \frac{\delta_i^{j+n}}{n} + \sum_{n=1}^{\infty} \frac{\delta_{i+n}^j}{n} + \Omega \delta_i^j, \quad (16)$$

for $1 \leq i \leq N, 1 \leq j < \infty$. The virtuality flow towards the IR and UV can be described using shift operators:

$$(\hat{\mathcal{S}}_{\text{IR}})_{i,j} = \delta_i^{j+1}, \quad (17)$$

$$(\hat{\mathcal{S}}_{\text{UV}})_{i,j} = \delta_{i+1}^j. \quad (18)$$

More formally, the Hamiltonian reads:

$$\hat{\mathcal{H}}_N = \sum_{n=1}^{N-1} \frac{(\hat{\mathcal{S}}_{\text{IR}})^n}{n} + \sum_{n=1}^{\infty} \frac{(\hat{\mathcal{S}}_{\text{UV}})^n}{n} + \Omega \hat{1} \quad (19)$$

$$= - \sum_{n=N}^{\infty} \frac{\hat{\mathcal{S}}_{\text{IR}}^n}{n} - \log(1 - \hat{\mathcal{S}}_{\text{IR}}) - \log(1 - \hat{\mathcal{S}}_{\text{UV}}) + \Omega \hat{1}. \quad (20)$$

In this framework the diffusion picture is clear. At $Y = 0$ the evolution starts with the initial condition $\varphi(Q^2, Y = 0) \sim \delta(Q^2 - Q_0^2)$, where $Q_0^2 = N_0 \Delta$. This corresponds to a single entry in the initial condition vector, *i.e.*

$$\vec{\phi}_0 \equiv (\phi_1^0, \phi_2^0, \dots, \phi_N^0, \dots)^t \quad (21)$$

with

$$\phi_i^0 = \frac{\delta_i^{N_0}}{\Delta}. \quad (22)$$

As each emission takes place at a rapidity strongly ordered with respect to the previous ones, it is possible to write the solution to Eq. (14) using the following formal expression

$$\vec{\phi} = e^{\alpha Y \hat{\mathcal{H}}_N} \cdot \vec{\phi}_0 \quad (23)$$

$$= \left\{ 1 + \int_0^Y dy_1 (\alpha \hat{\mathcal{H}}_N) + \int_0^Y dy_1 (\alpha \hat{\mathcal{H}}_N) \int_0^{y_1} dy_2 (\alpha \hat{\mathcal{H}}_N) \right. \\ \left. + \int_0^Y dy_1 (\alpha \hat{\mathcal{H}}_N) \int_0^{y_1} dy_2 (\alpha \hat{\mathcal{H}}_N) \int_0^{y_2} dy_3 (\alpha \hat{\mathcal{H}}_N) + \dots \right\} \cdot \vec{\phi}_0. \quad (24)$$

The action of the shift operators in the kernel displaces the original single “cell” in Eq. (22) towards lower or higher entries in the vector. This corresponds to the emission of real gluons in the s -channel, which modify the virtuality of the reggeized gluons “propagating” in the t -channel. The diagonal piece in the kernel, $\Omega \hat{1}$, does not modify the virtuality and corresponds to the “propagation” of a reggeized gluon with constant q^2 . Ω formally being an infinite quantity corresponds to a non-infrared-finite gluon Regge trajectory. Note that the solution to the original Eq. (11) is given by the last component of $\vec{\phi}$ in the continuum limit where $N \rightarrow \infty$, $\Delta \rightarrow 0$, while keeping $N\Delta = Q^2$ fixed.

3 $\text{sl}(2)$ spin chain Hamiltonian

As it was mentioned in the Introduction, there has been big progress in the understanding of the planar limit of the $\mathcal{N} = 4$ SYM dilatation operator by mapping it to the Hamiltonian of an integrable one-dimensional spin chain. The part of the theory of interest in this work is the non-compact bosonic $\text{sl}(2)$ closed subsector where states are constructed with scalar fields Φ (SO(6) Yang-Mills bosons) and their spacetime covariant derivatives $\mathcal{D}\Phi$, which scale under the $\text{sl}(2)$ subgroup of the Lorentz group. The corresponding charges are related to the SO(4,2) group and to the SO(6) \mathcal{R} -symmetry.

The commutation relations of the $\text{sl}(2)$ subalgebra of the superconformal algebra are $[J^{(+)}, J^{(-)}] = -2J^{(3)}$, $[J^{(3)}, J^{(\pm)}] = \pm J^{(\pm)}$. The spin chain Hamiltonian representing the one-loop anomalous dimensions of $\mathcal{N} = 4$ SYM operators with spin $S - 1$ in the planar limit of the $\text{sl}(2)$ sector reads

$$\mathcal{H}_{1,2}^{\text{sl}(2)} \theta(S - N) (a_1^\dagger)^{N-1} (a_2^\dagger)^{S-N} |00\rangle = -\lambda \sum_{l=1}^{\infty} \left(\frac{(1 - \delta_l^N)}{|l - N|} - (h(N - 1) + h(S - N)) \delta_l^N \right) \theta(S - l) (a_1^\dagger)^{l-1} (a_2^\dagger)^{S-l} |00\rangle, \quad (25)$$

where $\lambda = \frac{g^2 N_c}{8\pi^2}$ is the coupling and $h(N) = \sum_{l=1}^N \frac{1}{l} = \psi(N + 1) - \psi(1)$ is the harmonic number. The notation in terms of harmonic oscillators $(a^\dagger)^n |0\rangle = \frac{1}{n!} (\mathcal{D})^n \Phi$ (with $a|0\rangle = 0$, $[a, a^\dagger] = 1$), which corresponds to a site in a one-dimensional lattice (the total number of lattice sites is equal to the total \mathcal{R} -charge), has been used. n indicates that a given site is in the n -th excited state with respect to the $\text{Tr}(\Phi^2)$ vacuum. These excitations are classified in the spin $s = -\frac{1}{2}$ representation of $\text{sl}(2)$, which is infinite-dimensional. There is an overall trace in the operator cyclically ordering the different sites in the spin chain. Eq. (25) was introduced by Beisert in [25] and it corresponds to the nearest-neighbor one-loop Hamiltonian of an integrable XXX spin $s = -\frac{1}{2}$ chain. More explicitly, this Hamiltonian is invariant under the $\text{sl}(2)$ generators

$$J_{12}^{(+)} = a_1^\dagger (1 + a_1^\dagger a_1) + a_2^\dagger (1 + a_2^\dagger a_2), \quad (26)$$

$$J_{12}^{(-)} = a_1 + a_2, \quad (27)$$

$$J_{12}^{(3)} = 1 + a_1^\dagger a_1 + a_2^\dagger a_2. \quad (28)$$

It is also interesting its action on the following state:

$$\begin{aligned} \mathcal{H}_{1,2}^{\text{sl}(2)} (a_1^\dagger - a_2^\dagger)^\rho |00\rangle &= \sum_{s=0}^{\infty} (-1)^{\rho-s} \frac{\rho!}{s!(\rho-s)!} \mathcal{H}_{1,2}^{\text{sl}(2)} \theta(\rho-s) (a_1^\dagger)^s (a_2^\dagger)^{\rho-s} |00\rangle \\ &= 2h(\rho) (a_1^\dagger - a_2^\dagger)^\rho |00\rangle. \end{aligned} \quad (29)$$

4 The hidden BFKL / XXX $_{-\frac{1}{2}}$ spin chain mapping

To establish the mapping between the BFKL equation and the Hamiltonian associated to the dilatation operator for the $\mathfrak{sl}(2)$ sector of the $\mathcal{N} = 4$ SYM theory it is needed to treat the diagonal operator Ω in Eq. (12) of the BFKL kernel in a different way. The discretization of Eq. (11) can be also written in the form

$$\begin{aligned} \frac{\partial \phi_N}{\alpha \partial Y} &= \sum_{n=1}^{N-1} \frac{1}{N-n} \left(\phi_n - \frac{2n}{N+n} \phi_N \right) + \sum_{n=N+1}^{\infty} \frac{1}{n-N} \left(\phi_n - \frac{2N}{N+n} \phi_N \right) \\ &= \sum_{n=1}^{N-1} \frac{\phi_n}{N-n} + \sum_{n=N+1}^{\infty} \frac{\phi_n}{n-N} - 2h(N-1)\phi_N, \end{aligned} \quad (30)$$

This is a valid representation up to $\mathcal{O}\left(\frac{\phi_N}{N}\right)$ terms, which are negligible at large N , where the mapping will be defined. The difference between this representation and the previous one is in the regularization of the infrared divergence present in the gluon Regge trajectory. Different regularizations can be used which are equivalent in the $N \rightarrow \infty$ limit.

The new representation of the BFKL Hamiltonian now reads

$$\hat{\mathcal{H}}_N = - \sum_{n=N}^{\infty} \frac{\hat{S}_{\text{IR}}^n}{n} - \log(1 - \hat{S}_{\text{IR}}) - \log(1 - \hat{S}_{\text{UV}}) + \hat{\mathcal{G}}, \quad (31)$$

where

$$(\hat{\mathcal{G}})_{i,j} = -2h(i-1)\delta_i^j. \quad (32)$$

To connect with the XXX $_{-\frac{1}{2}}$ spin chain Hamiltonian acting on a state represented by a double copy of a harmonic oscillator it is needed to do this on a diagonal state with the same number of derivatives in each oscillator (both are in the same $(N-1)$ -th excited state):

$$\begin{aligned} \mathcal{H}_{1,2}^{\text{sl}(2)} \theta(N-1)(a_1^+)^{N-1}(a_2^+)^{N-1}|00\rangle &= \\ -\lambda \sum_{l=1}^{\infty} \left(\frac{(1-\delta_l^N)}{|l-N|} - 2h(N-1)\delta_l^N \right) \theta(2N-1-l)(a_1^+)^{l-1}(a_2^+)^{2N-1-l}|00\rangle, \end{aligned} \quad (33)$$

which is obtained by setting $S = 2N - 1$ in Eq. (25).

The equivalent Hamiltonian for the leading-order BFKL calculation in QCD and $\mathcal{N} = 4$ SYM theory can be found by simply rewriting the right hand side of Eq. (30) in the form

$$\mathcal{H}^{\text{BFKL}} \phi_N = \alpha \sum_{l=1}^{\infty} \left(\frac{(1-\delta_l^N)}{|l-N|} - 2h(N-1)\delta_l^N \right) \phi_l. \quad (34)$$

From the comparison of expressions (33) and (34) a natural mapping between the discretized BFKL gluon Green function and the double harmonic oscillator state emerges:

$$\phi_l \leftrightarrow \theta(2N-1-l)(a_1^+)^{l-1}(a_2^+)^{2N-1-l}|00\rangle. \quad (35)$$

To study how the convergence to the continuum $N \rightarrow \infty$ limit takes place it is useful to work with the function

$$\begin{aligned}\chi_N(\gamma) &= \sum_{l=1}^{\infty} \left(\frac{(1 - \delta_l^N)}{|l - N|} - 2h(N - 1)\delta_l^N \right) \left(\frac{N}{l} \right)^{\gamma} \\ &= \left(\sum_{l=1}^{N-1} - \sum_{l=N+1}^{\infty} \right) \frac{1}{N-l} \left(\frac{N}{l} \right)^{\gamma} - \sum_{l=1}^{N-1} \frac{2}{l}.\end{aligned}\quad (36)$$

In Fig.1 it is numerically shown that $\lim_{N \rightarrow \infty} \chi_N(\gamma) = \chi(\gamma)$ of Eq. (8) in the range $0 \leq \gamma \leq 1$. The convergence in N is not uniform in this region since it is much faster for small values of γ . Analytically, the continuum $N \rightarrow \infty$ limit can be found using $N = 1/\Delta$ and $l = x/\Delta$ with $\Delta \rightarrow 0$

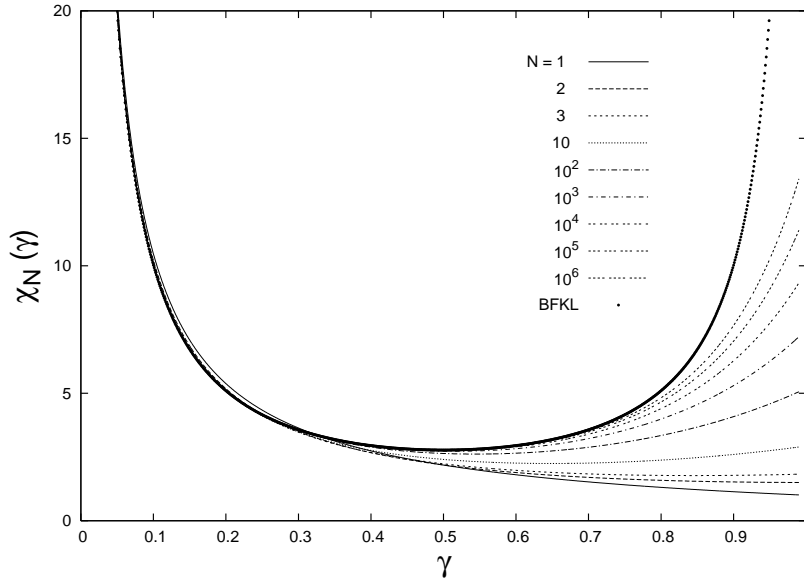


Figure 1: The function $\chi_N(\gamma)$ coincides with the BFKL eigenvalue $\chi(\gamma)$ for $\gamma \sim 0$ already with a low number of excitations N while it requires a very large number of them to match the $\gamma \sim 1$ region.

to obtain

$$\chi_{\infty}(\gamma) = \int_0^1 \frac{dx}{1-x} (x^{-\gamma} + x^{\gamma-1} - 2) = \chi(\gamma). \quad (37)$$

5 Discussion

In this Section possible consequences and open questions for the mapping between the BFKL gluon Green function and the double harmonic oscillator state are discussed in some detail. The interpre-

tation and answers to these are the bulk of current investigations.

5.1 Anomalous dimensions and wrapping contributions

The main assumptions in the asymptotic Bethe ansatz (ABA) were all-loop integrability and the existence of $\text{PSU}(2, 2|4)$ symmetry in the S-matrix. The ABA breaks down at order g^8 (four loops) for twist two operators due to wrapping corrections. The mapping proposed in this work should shed some light on the calculation of wrapping corrections in the spin chain context. A simple procedure would be to take the $N \rightarrow \infty$ limit, corresponding to an infinite number of magnons, in the action of the $\text{sl}(2)$ Hamiltonian on the states in the chain and repeat the steps which in the BFKL approach lead to the calculation of the leading $\mathcal{O}((\alpha/\omega)^n)$ contributions to the anomalous dimension of twist-two operators with spin $M = \omega - 1$, for $\omega \rightarrow 0$. To follow this path, Eq. (7) is written in the form

$$\varphi(Q^2, Y) = \int \frac{d\gamma}{2\pi i} \int \frac{d\omega}{2\pi i} \left(\frac{Q^2}{Q_0^2} \right)^{\gamma-1} \frac{e^{\omega Y}}{\omega - \alpha\chi_\infty(\gamma)}, \quad (38)$$

where $\chi_\infty(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1 - \gamma)$, and then notice that it is more convenient to perform first the integration in γ , *i.e.*

$$\varphi(Q^2, Y) \simeq \int \frac{d\omega}{2\pi i} \left(\frac{Q^2}{Q_0^2} \right)^{\gamma_\omega-1} e^{\omega Y}, \quad (39)$$

$$\omega = \alpha\chi_\infty(\gamma_\omega), \quad (40)$$

in order to extract the dominant contributions in the collinear region, which corresponds to the $Q^2 \gg Q_0^2$ limit. Eq. (40) then generates the anomalous dimension:

$$\gamma_\omega = \frac{\alpha}{\omega} + 2\zeta(3) \left(\frac{\alpha}{\omega} \right)^4 + 2\zeta(5) \left(\frac{\alpha}{\omega} \right)^6 + 12\zeta(3)^2 \left(\frac{\alpha}{\omega} \right)^7 + \dots \quad (41)$$

if the expansion

$$\omega = \alpha \left(\frac{1}{\gamma_\omega} + 2 \sum_{L=1}^{\infty} \zeta(2L+1) \gamma_\omega^{2L} \right) \quad (42)$$

of $\chi_\infty(\gamma_\omega)$ is used. It is only in the $N \rightarrow \infty$ limit that the complete expansion in terms of Riemann ζ functions appears and the correct wrapping terms in the anomalous dimension are found.

The common understanding is that it is in the long-range interactions of the spin chain where a mapping to higher-loop results in the gauge theory side should appear. It is in this sense that it would be very interesting to identify the correct variable in the $\text{sl}(2)$ spin chain dual to the rapidity Y in the BFKL framework and around which to generate a similar exponentiation of the Hamiltonian as in Eq. (38) and its asymptotic expansion at large Y in Eq. (9). This might lead to a simple method to obtain wrapping terms in the spin chain which could be also applied to Hamiltonians associated to other symmetries.

5.2 NLO kernel and comparison with other approaches

The mapping here discussed might also hold at higher loops both in the $\text{sl}(2)$ spin chain and in the BFKL formalism. The extension of Eq. (34) to next-to-leading order, *i.e.*, the discretization in

virtuality space of the azimuthal-angle averaged forward next-to-leading order BFKL equation in $\mathcal{N} = 4$ SYM can be written in the form

$$\begin{aligned} \mathcal{H}_{\text{NLO}}^{\text{BFKL}} \phi_N = & \alpha \sum_{l=1}^{\infty} \left\{ \frac{(1 - \delta_l^N)}{|l - N|} \left[1 - \frac{\alpha N}{N + l} \left(\zeta(2) + \left(1 - \frac{l}{N} \right) \sum_{m=1}^{\infty} \frac{1}{m^2} \left(\frac{N}{l} \right)^m \right) \right] \right. \\ & \left. + \delta_l^N \left[\mathcal{G}(N) \left(1 - \alpha \frac{\zeta(2)}{2} \right) + \alpha \frac{3}{2} \zeta(3) \right] \right\} \phi_l - \alpha^2 \sum_{l=1}^{N-1} \ln^2 \left(-\frac{l}{N} \right) \frac{\phi_l}{2(N + l)}. \end{aligned} \quad (43)$$

It will be worth to investigate the interplay between this formula and any higher order representation of the $\text{sl}(2)$ spin chain in future works, together with its interpretation in terms of wrapping contributions (the subleading in ω corrections for the anomalous dimension of Eq. (41) are found using Eq. (43)). Note that Eq. (43) would still correspond to a two-site spin chain with two states in their $N - 1$ excited level. At this point it is fair to bring into the discussion the results found by Shuvaev and Wallon in [27] where an expression as in Eq. (5) was found with φ corresponding there to the distribution of soft photons in a charged source, with no relation to high energy scattering in the Regge limit or the BFKL equation.

From a more mathematical point of view, it is likely that a link can be found between the matrix representation of the $\text{sl}(2)$ Hamiltonian here unveiled and the work of Adler and Van Moerbeke in [28], where it was investigated how the discrete sinh-Gordon equation leads to the Toeplitz 2-Toda lattice. This new lattice has τ -functions which are annihilated by operators living in a $\text{SL}(2, \mathbb{Z})$ subalgebra of the Virasoro algebra, and have a very similar structure to Eq. (19) if the harmonic weights $1/n$ are identified with time variables in the IR/UV directions. It is on these time variables where the Virasoro operators act.

To conclude, it is noteworthy that in [29] a coherent state representation for the Hamiltonian of the spin chain with $\text{sl}(2)$ symmetry was derived. The action of the Hamiltonian on the coherent states \vec{n}_1, \vec{n}_2 then reads

$$\langle \vec{n}_1, \vec{n}_2 | \mathcal{H}_{1,2}^{\text{sl}(2)} | \vec{n}_1, \vec{n}_2 \rangle = \log \left(1 - \frac{(\vec{n}_1 - \vec{n}_2)^2}{4} \right), \quad (44)$$

with $\vec{n}_{1,2}$ living on a two-dimensional hyperboloid for $\text{sl}(2)$. It will be interesting to study the relation with the Eq. (31) derived in this work.

6 Conclusions

We have unveiled a new mapping between the BFKL equation and the $s = -\frac{1}{2}$ XXX spin chain. This mapping requires of a simplification of the original BFKL equation to its forward limit and averaging over the azimuthal angle dependence of its kernel. The remaining physical variable corresponds to the virtuality of exchanged gluons. Discretization in this variable allows for a direct link between the gluon Green function, *i.e.*, the solution to the BFKL equation, and the two harmonic oscillators state representing the two-site $\text{sl}(2)$ spin chain. This is the main result of our analysis and it is presented in Eqs. (33,34,35). A new matrix representation of the BFKL equation has been given in Eq. (15), together with its physical interpretation in terms of a diffusion process into infrared and ultraviolet regions of virtuality space. We have proposed some directions for future research in the interpretation and better understanding of this mapping. These include the analysis of higher order corrections, the relation to Toeplitz lattices, the study of the mapping in terms of coherent states and how it could be related to the calculation of wrapping effects.

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